Root Systems and Root Lattices in Number Fields

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Online seminar
Quadratic Forms, Linear Algebraic Groups and Beyond
March 10, 2021
`https://qflag.org/`

This talk is based on the following papers joint with Yu. G. Zarhin:

[1] Vladimir L. Popov, Yuri G. Zarhin, *Root systems in number fields*, Indiana University Mathematics Journal **70** (2021), no. 1, 285–300.

[2] Vladimir L. Popov, Yuri G. Zarhin, *Root lattices in number fields*, Bulletin of Mathematical Sciences (2020), `https://doi.org/10.1142/S1664360720500216`.

[3] V. L. Popov, Yu. G. Zarhin, *Rings of integers in number fields, and root lattices*, Doklady Mathematics **101** (2020), no. 3, 221–223.

Starting point

Construction of a root system $\mathbf{G_2}$:

J-P. Serre, ***Lie Algèbres de Lie Semi-simples Complexes*****, Benjamin, New York, 1966, §16:**

"This system can be described as the set of algebraic integers of a cyclotomic field generated by a cubic root of unity, with the norm 1 or 3."

Part 1: Realization of root systems in number fields

Root systems: reminder

Let V be a finite-dimensional vector space over $\mathbb{Q}$ and let $v \in V$ be a nonzero vector.

A linear map $\varrho\colon V \to V$ is called a **reflection with respect to v** if

- $\varrho(v) = -v$,
- V^{ϱ} is a hyperplane in V.

In this case, for the linear operator $\varrho - \mathrm{id}$,

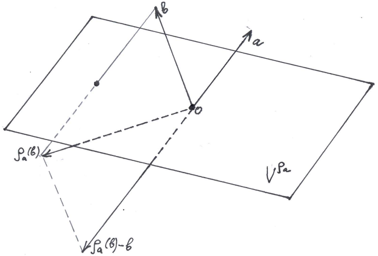
- the *image* of $\varrho - \mathrm{id}$ is the line $\mathbb{Q}v$,
- the *kernel* of $\varrho - \mathrm{id}$ is the hyperplane V^{ϱ}.

Root systems: reminder

Definition

Let V be the $\mathbb{Q}$-linear span of a finite set R and $0 \notin R$. If the following hold, then R is called a <u>**root system in V**</u>:

- for every $a \in R$, there is a reflection ϱ_a with respect to a such that $\varrho_a(R) = R$ (such a ϱ_a is automatically unique);
- $(\varrho_a - \mathrm{id})(b) \in \mathbb{Z}a$ for all $a, b \in R$.



Root systems: reminder

Properties and terminology:

Let R be a root system in V.

• The $\mathbb{Z}$-linear span of R in V is a free $\mathbb{Z}$-module of rank $\dim V$. Its rank is called the **rank of R**.

• The group $W(R)$ generated by all reflections ϱ_a, $a \in R$, is finite and called the **Weyl group** of the root system R.

The type of Dynkin diagram of R is called the **type of R.**

Let L be a free $\mathbb{Z}$-module of finite rank $n > 0$ considered as a subset of the $\mathbb{Q}$-vector space $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$.

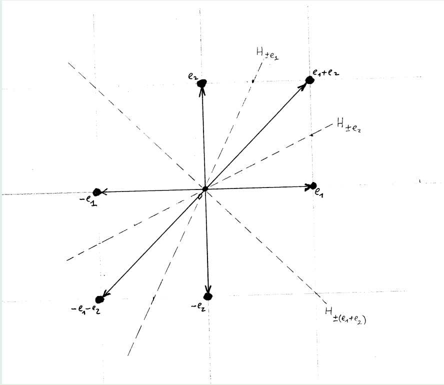
Every type R of root systems of rank n is realizable in L:

there is a root system R in V of type R such that $R \subset L$.

However, if the pair (V, L) is endowed with an additional structure, then the Weyl $W(R)$ may not be consistent with it. For instance, if V is endowed with an inner product, then $W(R)$ may contain nonorthogonal transformations.

Example

Let $n = 2$ and let $e_1, e_2 \in L$ be an orthonormal basis in V. Then $R = \{\pm e_1, \pm e_2, \pm(e_1 + e_2)\}$ is the root system in V of type A_2. Not all transformations from the Weyl group $W(R)$ are orthogonal.



A natural source of pairs (V, L) is **algebraic number theory**, in which they arise in the form $(K, \mathscr{O})$, where **K is a number field** and **$\mathscr{O}$ is the ring of integers of K**.

Some additional structures/objects are naturally associated with every pair $(K, \mathscr{O})$. Among them are the following **three subgroups in $\mathrm{GL}_{\mathbb{Q}}(K)$**:

- the automorphism group $\mathrm{Aut}(K)$ of the field K;

- the group $\mathrm{mult}(K^*)$, where $\mathrm{mult}(a)$ is the operator of multiplication by $a \in K^*$:

$$\mathrm{mult}(a)\colon\ K \to K,\ x \mapsto ax.$$

- the group $\mathcal{L}(K)$ generated by $\mathrm{Aut}(K)$ and $\mathrm{mult}(K^*)$.

Realizations of a root system type in a number field

Definition

We say that a type R of (not necessarily reduced) root systems **admits a realization in a number field** K, if

- $[K:\mathbb{Q}] = \mathrm{rk}(\mathrm{R})$;

- there is a subset R of rank $\mathrm{rk}(\mathrm{R})$ in $\mathscr{O}$, which is a root system of type R such that $W(R)$ is a subgroup of the group $\mathcal{L}(K)$.

In this case, R is called a **realization of the type** R **in the field** K.

Remark

In this definition, replacing $\mathscr{O}$ by K does not yield a broader concept

Explanation:

In this case, there is a nonzero $m \in \mathbb{Z}$ such that

$$m \cdot R := \{m\alpha \mid \alpha \in R\} \subset \mathscr{O}.$$

The set $m \cdot R$ has rank $\mathrm{rk}(\mathrm{R})$, it is a root system in K of type R, and $W(m \cdot R) = W(R)$.

Integer elements of a fixed norm

Notation:

$\mathscr{O}(d)$ is the set of all elements of $\mathscr{O}$, whose norm is d.

Realizations of rank 1 root system types in number fields

Root systems of types A_1 and BC_1:

Take $K = \mathbb{Q}$. Then $\mathscr{O} = \mathbb{Z}$ and $\mathcal{L}(K) = \mathrm{mult}(\mathbb{Q}^*)$.

Let $\alpha \in \mathbb{Z}$, $\alpha \neq 0$. Then

$$R := \{\pm\alpha\} \text{ и } R := \{\pm\alpha, \pm 2\alpha\}$$

are the realizations of types A_1 and BC_1 in the field K.

Root systems of types $\mathbf{A_2}$ and $\mathbf{G_2}$:

Let K be the third cyclotomic field:

$$K = \mathbb{Q}(\sqrt{-3}).$$

Then $\mathscr{O} = \mathbb{Z} + \mathbb{Z}\omega$, where $\omega = e^{2\pi i/6} = (1 + i\sqrt{3})/2$, and $\mathrm{Aut}(K) = \langle c \rangle$, where c is the complex conjugation $a \mapsto \overline{a}$.

Every element $a \in \mathcal{L}(K)$ of a finite order is the orthogonal transformation $a : K \to K$ with respect to the Euclidean structure on K:

$$K \times K \to \mathbb{Q},\ (a, b) \mapsto \mathrm{Trace}_{K/\mathbb{Q}}(a\overline{b}) = 2\mathrm{Re}(a\overline{b}),$$

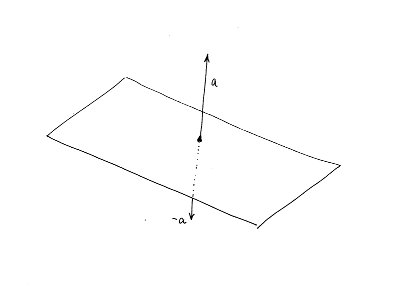
Realizations of rank 2 root system types in number fields

Example

For every nonzero element $a \in K$, the operator

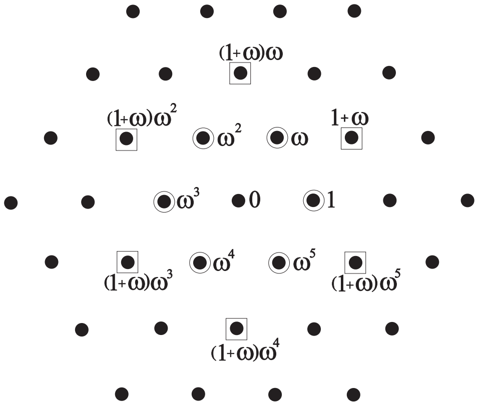
$$r_a := \mathrm{mult}(-a\overline{a}^{-1})c \in \mathcal{L}(K)$$

is a reflection with respect to a.



Realizations of rank 2 root system types in number fields

$\mathscr{O}(1) = \{\pm 1, \pm\omega, \pm\omega^2\}$ (all 6th roots of 1).
$\mathscr{O}(3) = (1+\omega)\mathscr{O}(1)$.



Realizations of rank 2 root system types in number fields

$$\mathscr{O}(1) = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\} \text{ where } \alpha_1 = 1,\ \alpha_2 = \omega^2.$$

Therefore,

- $\mathscr{O}(1)$ is the root system in K of type A_2 with the base α_1, α_2.
- $\mathscr{O}(3)$ is the root system in K of type A_2 with the base $\beta_1 = (1+\omega)\alpha_1$, $\beta_2 = (1+\omega)\alpha_2$.
- $\mathscr{O}(1) \bigcup \mathscr{O}(3)$

$$= \{\pm\alpha_1, \pm\beta_2, \pm(\alpha_1+\beta_2), \pm(2\alpha_1+\beta_2), \pm(3\alpha_1+\beta_2), \pm(3\alpha_1+2\beta_2)\}$$

is the root system in K of type G_2 with the base α_1, β_2.

Realizations of rank 2 root system types in number fields

For every $a \in \mathscr{O}(1) \bigcup \mathscr{O}(3)$ and positive integer d,

$$r_a(\mathscr{O}(d)) = \mathscr{O}(d).$$

Therefore, $W(\mathscr{O}(1))$, $W(\mathscr{O}(3))$, and $W(\mathscr{O}(1) \bigcup \mathscr{O}(3))$ are the subgroups of $\mathcal{L}(K)$. Hence

- $\mathscr{O}(1)$ is the realization of type A_2 in the field K,
- $\mathscr{O}(3)$ is the realization of type A_2 in the field K,
- $\mathscr{O}(1) \bigcup \mathscr{O}(3)$ is the realization of type G_2 in the field K

Realizations of rank 2 root system types in number fields

Root systems of types $\mathbf{B}_2$, $\mathbf{2A}_1$, $\mathbf{BC}_2$, $\mathbf{2BC}_1$, and $\mathbf{A}_1 \dot{+} \mathbf{BC}_1$:

Let K be the fourth cyclotomic field:

$$K = \mathbb{Q}(\sqrt{-1}).$$

Then $\mathscr{O} = \mathbb{Z} + \mathbb{Z}i$ and $\mathrm{Aut}(K) = \langle c \rangle$, where c is the complex conjugation $a \mapsto \overline{a}$.

Every element $a \in \mathcal{L}(K)$ of a finite order is the orthogonal transformation $a : K \to K$ with respect to the same Euclidean structure on K as above:

$$K \times K \to \mathbb{Q},\ (a, b) \mapsto \mathrm{Trace}_{K/\mathbb{Q}}(a\overline{b}) = 2\mathrm{Re}(a\overline{b}),$$

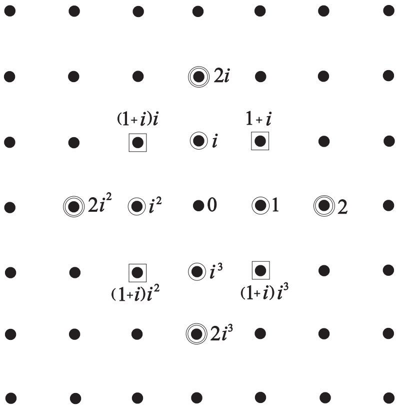
As above, for every nonzero element $a \in K$, the operator

$$r_a := \mathrm{mult}(-a\overline{a}^{-1})c \in \mathcal{L}(K)$$

is a reflectrion with respect to a.

Realizations of rank 2 root system types in number fields

$\mathscr{O}(1) = \{\pm 1, \pm i\}$ (all 4th of 1).
$\mathscr{O}(2) = (1+i)\mathscr{O}(1)$.
$\mathscr{O}(4) = 2\mathscr{O}(1)$.



Realizations of rank 2 root system types in number fields

$\mathscr{O}(1) = \{\pm\alpha_1, \pm\alpha_2\}$ where $\alpha_1 = 1$, $\alpha_2 = i$.

Therefore,

• $\mathscr{O}(1)$, $\mathscr{O}(2)$, $\mathscr{O}(4)$ are the root systems in K of type $\mathrm{A}_1 \dot{+} \mathrm{A}_1$ resp. with the base

$$\alpha_1, \alpha_2, \quad \beta_1 = (1+i)\alpha_1, \beta_2 = (1+i)\alpha_2, \quad \text{and} \quad 2\alpha_1,\ 2\alpha_2.$$

• $\mathscr{O}(1) \bigcup \mathscr{O}(2) = \{\pm\alpha_1, \pm\beta_2, \pm(\alpha_1+\beta_2), \pm(2\alpha_1+\beta_2)\}$, is the root system in K of type B_2 with the base α_1, β_2.

Realizations of rank 2 root system types in number fields

- $\mathscr{O}(1)\bigcup\mathscr{O}(4)$ is the root system in K of type $2\mathsf{BC}_1$ with the base α_1, α_2.

- $\mathscr{O}(1)\bigcup\{\pm 2\}$ is the root system in K of type $\mathsf{A}_1 \dotplus \mathsf{BC}_1$ with the base α_1, α_2.

- $\mathscr{O}(1)\bigcup\mathscr{O}(2)\bigcup\mathscr{O}(4)$
$$= \{\pm\alpha_1, \pm 2\alpha_1, \pm\beta_2, \pm(\alpha_1+\beta_2), \pm 2(\alpha_1+\beta_2), \pm(2\alpha_1+\beta_2)\},$$

is the root system in K of type BC_2 with the base α_1, β_2.

Realizations of rank 2 root system types in number fields

For every $a \in \mathscr{O}(1) \bigcup \mathscr{O}(2)$ and positive integer d,

$$r_a(\mathscr{O}(d)) = \mathscr{O}(d).$$

Therefore, $W(\mathscr{O}(1))$, $W(\mathscr{O}(1)) \bigcup W(\mathscr{O}(2))$, $W(\mathscr{O}(1) \bigcup \mathscr{O}(4))$, $W(\mathscr{O}(1) \bigcup \mathscr{O}(2) \bigcup \mathscr{O}(4))$, $W(\mathscr{O}(1) \bigcup \{\pm 2\})$ are the subgroups of $\mathcal{L}(K)$. Hence

- $\mathscr{O}(1)$ is the realization of type $\mathsf{A}_1 \dot{+} \mathsf{A}_1$ in the field K,
- $\mathscr{O}(1) \bigcup \mathscr{O}(2)$ is the realizations of type B_2 in the field K,
- $\mathscr{O}(1) \bigcup \mathscr{O}(4)$ is the realizations of type $\mathsf{BC}_1 \dot{+} \mathsf{BC}_1$ in the field K,
- $\mathscr{O}(1) \bigcup \mathscr{O}(2) \bigcup \mathscr{O}(4)$ is therealizations of type BC_2 in the field K,
- $\mathscr{O}(1) \bigcup \{\pm 2\}$ is the realizations of type $\mathsf{A}_1 \dot{+} \mathsf{BC}_1$ in the field K

Arbitrary rank case

Theorem

The following properties of the Weyl group of a reduced root system of type R *and rank* n *are equivalent:*

- *This Weyl group is isomorphic to a subgroup of the group* $\mathcal{L}(K)$*, where* K *is a number field of degree* n *over* $\mathbb{Q}$*.*
- R *is contained in the following list:*

$$\mathsf{A}_1,\quad \mathsf{A}_2,\quad \mathsf{B}_2,\quad \mathsf{G}_2,\quad \mathsf{A}_1 \dotplus \mathsf{A}_1,\quad \mathsf{A}_1 \dotplus \mathsf{A}_1 \dotplus \mathsf{A}_2,\quad \mathsf{A}_2 \dotplus \mathsf{B}_2.$$

Comparing this theorem with the next one shows the following:

The existence of an isomorphism between a subgroup G of the group $\mathcal{L}(K)$ and the Weyl group of a root system of rank $[K : \mathbb{Q}]$ and type R **is not equivalent** to the fact that $G = W(R)$, where R is a root system of type R in $\mathscr{O}$.

Classification of root systems types realizable in numeric fields

Theorem

For every type R *of root systems (not necessarily reduced), the following properties are* *equivalent:*

- *There is a number field, in which* R *admits a realization;*
- $\mathrm{rk}(\mathrm{R}) = 1$ *or* 2.

Part 2: Realizations of root lattices in number fields

Lattices

Definition

We call a **lattice** every pair (L, b), where L is a free $\mathbb{Z}$-module of a finite rank, and

$$b\colon L \times L \to \mathbb{Z}$$

is a *nondegerate symmetric* bilinear form.

In what follows, L is always considered as an additive subgroup in the vector space $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$ над $\mathbb{Q}$.

Definitions

Definition

A nonzero lattice (L, b) is called **primitive**, if the greatest common divisor of all integers $b(x, y)$, where $x, y \in L$, equals 1.

Definition

A lattice (L, b) is called **even** if $b(x, x) \in 2\mathbb{Z}$ for all $x \in L$.

Definitions

Definition

A lattice is called **indecomposable** if it is inexpressible as orthogonal direct sum of nonzero sublattices

Definition

A lattice (L_1, b_1) is called **similar** to a lattice (L_2, b_2) if there are integers $m_1, m_2 \in \mathbb{Z}$ such that $(L_1, m_1 b_1)$ and $(L_2, m_2 b_2)$ are isometric.

Notation

The orthogonal direct sum of s copies of a lattice (L, b) is denoted by $(L, b)^s$.

Root lattices

Definition

A nonzero lattice is called a **root lattice** if it is isometric to orthogonal direct sum of lattices belonging to the union of two infinite series $\mathbb{A}_\ell$ $(\ell \geqslant 1)$, $\mathbb{D}_\ell$ $(\ell \geqslant 4)$, and four sporadic lattices $\mathbb{Z}^1$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, whose explicit description is given below.

Notation

• $\mathbb{R}^m$ is the m-dimensional coordinate real vector space of rows endowed with the standard Euclidean structure

$$\mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}, \quad ((x_1, \ldots, x_m), (y_1, \ldots, y_m)) := \textstyle\sum_{j=1}^m x_j y_j. \quad (*)$$

• $e_j := (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is on the jth position.

• If L is the $\mathbb{Z}$-linear span of a set of linearly independent elements of $\mathbb{R}^m$ such that $b(L \times L) \subseteq \mathbb{Z}$, where b is the restriction of map $(*)$ to $L \times L$, then (L, b) is called a lattice in $\mathbb{R}^m$ and denoted just by L.

Then, using these notation and conventions,

- $\mathbb{Z}^n$ is the lattice $\{(x_1, \dots, x_n) \mid x_j \in \mathbb{Z} \text{ for all } j\}$ in $\mathbb{R}^n$.
- $\mathbb{A}_n$ is the lattice $\{(x_1, \dots, x_{n+1}) \in \mathbb{Z}^{n+1} \mid \sum_{j=1}^{n+1} x_j = 0\}$ in $\mathbb{R}^{n+1}$.
- $\mathbb{D}_n$ is the lattice $\{(x_1, \dots, x_n) \in \mathbb{Z}^n \mid \sum_{j=1}^{n} x_j \text{ is even}\}$ in $\mathbb{R}^n$, $n \geqslant 4$.
- $\mathbb{E}_8$ is the lattice $\mathbb{D}_8 \bigcup \left(\mathbb{D}_8 + \frac{1}{2}(e_1 + \cdots + e_8)\right)$ in $\mathbb{R}^8$.
- $\mathbb{E}_7$ is the orthogonal in $\mathbb{E}_8$ of the sublattice $\mathbb{Z}(e_7 + e_8)$.
- $\mathbb{E}_6$ is the orthogonal in $\mathbb{E}_8$ of the sublattice $\mathbb{Z}(e_7 + e_8) + \mathbb{Z}(e_6 + e_8)$.

Properties of lattices $\mathbb{A}_\ell$, $\mathbb{D}_\ell$, $\mathbb{Z}^1$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$

- $\mathbb{A}_\ell$, $\mathbb{D}_\ell$, $\mathbb{Z}^1$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ are indecomposable.

- Decomposition of any root lattice as orthogonal direct sum of indecomposable lattices (called its **indecomposable components**) is unique.

- $\mathbb{A}_\ell$ при $\ell \neq 1$, $\mathbb{D}_\ell$, $\mathbb{Z}^1$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ are primitive.

- $\mathbb{A}_1$ is not primitive.

- $\mathbb{A}_\ell$, $\mathbb{D}_\ell$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ are even.

- $\mathbb{Z}^1$ is not even.

Root lattices and root systems

• If R is a root system in a vector space V over $\mathbb{Q}$, and $L = \mathbb{Z}R$, then there is bilinear form $b\colon L \times L \to \mathbb{Z}$ such that (L, b) is a root lattice.

• Every root lattice is obtained in this fashion (generally speaking, not in the only way).

• If R is irreducible, then in all cases except type $\mathbb{A}_1$, the bilinear form b is uniquely determined by R and the set of following four conditions:

(a) b is invariant with respect to the Weyl group $W(R)$,
(b) b takes values in $\mathbb{Z}$,
(c) b is positive-definite.
(d) (L, b) is primitive.

Root lattices and root systems

For an irreducible reduced root system R, the relationship between the type of R and the type of the root lattice (L, b) is given by the following table:

type of R	type of (L, b)
A_ℓ, $\ell \geqslant 1$	$\mathbb{A}_\ell$
B_ℓ, $\ell \geqslant 2$	$\mathbb{Z}^\ell$
C_ℓ, $\ell \geqslant 3$	$\mathbb{D}_\ell$
C_2	$\mathbb{Z}^2$
D_ℓ, $\ell \geqslant 3$	$\mathbb{D}_\ell$
E_ℓ, $\ell = 6, 7, 8$	$\mathbb{E}_\ell$
F_4	$\mathbb{D}_4$
G_2	$\mathbb{A}_2$

Theorem (E. Witt)

A lattice (L, b) is a root lattice if and only if the following two conditions hold:

(i) *the form b is positive-definite;*

(ii) *the $\mathbb{Z}$-module L is generated by the set*

$$\{x \in L \mid b(x,x) = 1 \text{ or } 2\}.$$

In view of Witt's theorem, all root lattices are split into the following **three disjoint types**:

Three types of root lattices

- **Root lattices of unmixed type** I

These are the lattices (L, b), for which the $\mathbb{Z}$-module L is generated by the set $\{x \in L \mid b(x, x) = 1\}$.

Equivalent description :

These are exactly all lattices isometric to $\mathbb{Z}^n$.

Three types of root lattices

- **Root lattices of unmixed type** II

These are the lattices (L, b), for which the $\mathbb{Z}$-module L is generated by the set $\{x \in L \mid b(x,x) = 2\}$.

Equivalent description:

These are exactly all even root lattices.

One more equivalent description:

These are exactly all root lattices, all of whose indecomposable components are not isometric to $\mathbb{Z}^1$.

Three types of root lattices

- **Root lattices of mixed type**

These are all other root lattices.

Constructions of lattices in number fields

Algebraic number theory is a natural source of lattices. Namely:

Let K be a number field, let $\mathscr{O}$ be the ring of integers of K, and

$$n := [K : \mathbb{Q}] < \infty.$$

Let $\sigma_1, \ldots, \sigma_n$ be the set of all field embeddings $K \hookrightarrow \mathbb{C}$.

A classical construction of geometric representation of algebraic numbers embeds K into the space $\mathbb{R}^n$ endowed with the standard Euclidean structure. This endows K (and hence $\mathscr{O}$) with the following $\mathbb{Q}$-bilinear form:

$$b_K\colon K \times K \to \mathbb{C}, \quad b_K(x,y) := \textstyle\sum_{j=1}^n \sigma_j(x)\overline{\sigma_j(y)}.$$

Constructions of lattices in number fields

Theorem

- *The $\mathbb{Q}$-linear span of the set $b_K(K \times K)$ is a proper subset of $\mathbb{R}$ containing $\mathbb{Q}$.*
- *The bilinear form b_K is symmetric and positive-definite.*
- *Properties (a), (b), (c) listed below are equivalent:*
  - (a) $b_K(K \times K) = \mathbb{Q}$.
  - (b) $b_K(\mathscr{O} \times \mathscr{O}) \subseteq \mathbb{Q}$.
  - (c) *There is $\tau \in \operatorname{Aut} K$ such that $\tau^2 = \operatorname{id}$ and*

$$b_K(x, y) = \operatorname{Trace}_{K/\mathbb{Q}}(x \cdot \tau(y)) \quad \text{for all } x, y \in K.$$

- *If (c) holds, then either K is totally real and $\tau = \operatorname{id}$ or K is a CM-field and τ is the complex conjugation.*

Constructions of lattices in number fields

Generalization of the classical construction

We fix an involutive automorphism

$$\theta \in \operatorname{Aut} K, \quad \theta^2 = \mathrm{id}.$$

Then

$$\mathrm{tr}_{K,\theta}\colon K \times K \to \mathbb{Q}, \quad \mathrm{tr}_{K,\theta}(x, y) := \mathrm{Trace}_{K/\mathbb{Q}}(x \cdot \theta(y))$$

is a nondegenerate symmetric bilinear form and for every nonzero ideal I in $\mathscr{O}$, the pair

$$(I, \mathrm{tr}_{K,\theta}) := (I, \mathrm{tr}_{K,\theta}|_{I \times I})$$

is a lattice of rank n.

Constructions of lattices in number fields

Further generalization

Let J be a nonzero (fractional) ideal of K, and let $a \in K$ be a nonzero element such that $\theta(a) = a$, $\mathrm{Trace}_{K/\mathbb{Q}}(ax \cdot \theta(y)) \in \mathbb{Z}$ for all $x, y \in J$. Let

$$\mathrm{tr}_{K,\theta,J,a}\colon J \times J \to \mathbb{Z}, \quad \mathrm{tr}_{K,\theta,J,a}(x, y) := \mathrm{Trace}_{K/\mathbb{Q}}(ax \cdot \theta(y)).$$

Then $(J, \mathrm{tr}_{K,\theta,J,a})$ is a lattice of rank n.

The origins of this construction essentially go back to Gauss:

for $n = 2$ and $a = 1/\mathrm{Norm}_{K/\mathbb{Q}}(J)$ it gives the classical correspondence between ideals and quadratic binary forms established by Gauss.

Remarkable lattices of the form $(J, \mathrm{tr}_{K,\theta,J,a})$

Some remarkable lattices are isometric to lattices of the form $(J, \mathrm{tr}_{K,\theta,J,a})$.

Examples, in which K is a dth cyclotomic field

- Root lattices $\mathbb{A}_{p-1}$ with odd prime p for $d = p$ (Ebeling).
- Root lattices $\mathbb{E}_6$ and $\mathbb{E}_8$, for $d = 9$ and resp. $d = 15, 20, 24$ (Bayer-Fluckiger).
- Coxeter–Todd lattice for $d = 21$ (Bayer-Fluckiger, Martinet).
- Leech lattice for $d = 35, 39, 52, 56, 84$ (Bayer-Fluckiger, Quebbemann).
- Classification of root lattices isometric to $(J, \mathrm{tr}_{K,\theta,J,a})$ type lattices is known (Bayer-Fluckiger, Martinet).

Problems

- *Given a lattice* (L, b), *find out whether it is isometric to a lattice of the form* $(J, \mathrm{tr}_{K,\theta,J,a})$ *for suitable* K, θ, J, a.
- *Given a lattice* (L, b), *a field* K, *and its nonzero ideal* J, *find out if there are* θ *and* a *such that* $(J, \mathrm{tr}_{K,\theta,J,a})$ *and* (L, b) *are isometric lattices.*

Among all nonzero ideals of K there is a naturally distinguished one, namely, $\mathscr{O}$. For it, there is a naturally distinguished a suitable for all automorphisms θ, namely, $a = 1$.

This leads to the **problem of finding remarkable lattices isometric (or, more generally, similar) to lattices of the form $(\mathscr{O}, \mathrm{tr}_{K,\theta})$**.

Below this problem is considered for root lattices.

Problems

Problems

(**R**) *Classify pairs K, θ, for which $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ is a root lattice.*

(**S**) *Generalization: Classify pairs K, θ, for which $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ is similar to a root lattice.*

The following examples show that pairs K, θ with the indicated properties do exist.

Example

Let $n = 1$.
Тогда $K = \mathbb{Q}$, $\mathscr{O} = \mathbb{Z}$, $\theta = \mathrm{id}$, and $\mathrm{Trace}_{K/\mathbb{Q}}(x) = x$ for all $x \in K$.
Therefore, in this case, $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ is the root lattice $\mathbb{Z}^1$ (which is similar but not isometric to the lattice $\mathbb{A}_1$).

Examples of root lattices of the form $(\mathscr{O}, \mathrm{tr}_{K,\theta})$

Example

Let $n = 2$ and let K be the 3rd cyclotomic field: $K = \mathbb{Q}(\sqrt{-3})$. Let θ be the complex conjugation. Then $\mathscr{O} = \mathbb{Z} + \mathbb{Z}\omega$, where $\omega = (1 + \sqrt{-3})/2$, and

$$\mathrm{Trace}_{K/\mathbb{Q}}(x) = x + \theta(x) = 2\mathrm{Re}(x) \text{ for all } x \in K.$$

Therefore, $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ is the root lattice isometric to $\mathbb{A}_2$.

Example

Let $n = 2$ and let K be the 4th cyclotomic field: $K = \mathbb{Q}(\sqrt{-1})$. Let θ be the complex conjugation. Then $\mathscr{O} = \mathbb{Z} + \mathbb{Z}\sqrt{-1}$ and

$$\mathrm{Trace}_{K/\mathbb{Q}}(x) = x + \theta(x) = 2\mathrm{Re}(x) \text{ for all } x \in K.$$

Therefore, $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ is the root lattice isometric to $\mathbb{A}_1^2$.

Solution to Problem (R):

Theorem

The following properties of a pair K, θ are equivalent:

- $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ *is a root lattice;*
- K, θ *is one of the following three pairs:*
  - $K = \mathbb{Q}$, $\theta = \mathrm{id}$;
  - $K = \mathbb{Q}(\sqrt{-3})$, θ *is the complex conjugation;*
  - $K = \mathbb{Q}(\sqrt{-1})$, θ *is the complex conjugation.*

Problem (S)

Let us now consider problem (S) on the classification of lattices $(\mathscr{O}, \mathrm{tr}_{K,\theta})$, which are similar (but not necessarily isometric) to root lattices. It appears that there are many more of them, than the lattices $(\mathscr{O}, \mathrm{tr}_{K,\theta})$, which are root ones.

Notation:

m is the unique positive integer such that

$$\mathrm{Trace}_{K/\mathbb{Q}}(\mathscr{O}) = m\mathbb{Z}$$

(such m exists because $\mathrm{Trace}_{K/\mathbb{Q}}\colon \mathscr{O} \to \mathbb{Z}$ is a nonzero additive group homomorphism).

Classification of lattices $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ similar to root lattices of unmixed type I

Theorem

The following properties of a pair K, θ are equivalent:

(a) $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ *is similar to* $\mathbb{Z}^n$;

(b) $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ *is similar to* $\mathbb{A}_1^n$;

(c) $(\mathscr{O}, \mathrm{tr}_{K,\theta}/m)$ *is isometric to* $\mathbb{Z}^n$;

(d) $(\mathscr{O}, 2\,\mathrm{tr}_{K,\theta}/m)$ *is isometric to* $\mathbb{A}_1^n$;

(e) K *is a* 2^a*th cyclotomic field, where* $a \in \mathbb{Z}$, $a > 0$, *and* θ *is the complex conjugation if* $a > 1$, *and* $\theta = \mathrm{id}$ *if* $a = 1$.

If these properties hold, then $n = 2^{a-1}$ *and* $m = n$.

In (e), *let* $\zeta_{2^a} \in K$ *be a* 2^a*th primitive root of* 1, *and let* $x_j := \zeta_{2^a}^j$. *Then the set of all indecomposable components of the root lattice* $(\mathscr{O}, \mathrm{tr}_{K,\theta}/m)$ *coincides with the set of all its sublattices* $\mathbb{Z}x_j$, $0 \leqslant j \leqslant 2^{a-1} - 1$. *For every* j, *the value of* $\mathrm{tr}_{K,\theta}/m$ *at* (x_j, x_j) *is* 1.

Classification of lattices $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ similar to root lattices of unmixed type II

Theorem

The following properties of a pair K, θ are equivalent:

(a) $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ *is similar to an even primitive root lattice.*

(b) $(\mathscr{O}, \mathrm{tr}_{K,\theta}/m)$ *is an even primitive root lattice.*

(c) *n is even and $(\mathscr{O}, \mathrm{tr}_{K,\theta}/m)$ is isometric to $\mathbb{A}_2^{n/2}$.*

(d) *K is a 2^a3^bth cyclotomic field, where $a, b \in \mathbb{Z}$, $a > 0$, $b > 0$, and θ is the complex conjugation.*

If these properties hold, then $n = 2^a3^{b-1}$ and $m = n/2$.

In (d), let ζ_{2^a} and $\zeta_{3^b} \in K$ be respectively a primitive 2^ath and $3^b$the roots of 1. Let $x_{i,j} := \zeta_{2^a}^i \zeta_{3^b}^j$. Then the set of all indecomposable components of the root lattice $(\mathscr{O}, \mathrm{tr}_{K,\theta}/m)$ coincides with the set of all its sublattices $\mathbb{Z}x_{i,j} + \mathbb{Z}x_{i,j+3^{b-1}}$, $0 \leqslant i \leqslant 2^{a-1} - 1$, $0 \leqslant j \leqslant 3^{b-1} - 1$. For all i, j, the values of $\mathrm{tr}_{K,\theta}/m$ at $(x_{i,j}, x_{i,j})$, $(x_{i,j+3^{b-1}}, x_{i,j+3^{b-1}})$, and $(x_{i,j}, x_{i,j+3^{b-1}})$ are, respectively, 2, 2 and -1.

Application: $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ and the Leech lattice

Since $\mathbb{E}_8$ is the unique (up to isometry) positive-definite even unimodular lattice of rank 8, as a corollary of the previous theorem, we obtain

Theorem

Every lattice $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ is not similar to the Leech lattice.

Application: $(\mathscr{O}, \operatorname{tr}_{K,\theta})$ and positive-definite even unimodular lattices

In fact, we obtain a more general result:

Theorem

Every positive-definite even unimodular lattice of rank $\leqslant 48$
not similar to a lattice of the form $(\mathscr{O}, \operatorname{tr}_{K,\theta})$.

This theorem excludes many lattices from being similar to lattices of the form $(\mathscr{O}, \operatorname{tr}_{K,\theta})$. Indeed, if $\Phi(r)$ is the number of pairwise nonisometric positive-definite even unimodular lattices of rank r, then

$$\Phi(8) = 1,\ \Phi(16) = 2,\ \Phi(24) = 24,\ \Phi(32) \geqslant 10^7,\ \Phi(48) \geqslant 10^{51}.$$

Lattices $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ similar to root lattices of mixed type

Notation:

μ_K is the (finite cyclic) multiplicative group of all roots of 1 in K.

$\bigoplus$ denotes the orthogonal direct sum of lattices.

Restrictions on lattices $(\mathscr{O}, \mathrm{tr}_{K,\theta})$, similar to root lattices of mixed type

Theorem

If $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ is similar to a root lattice of mixed type, then

(a) $m = n > 1$;

(b) *all prime numbers dividing the number n are ramified in the field extension $K/\mathbb{Q}$ and if a prime $p \in \mathbb{Z}$ is ramified in $K/\mathbb{Q}$, then $p \leqslant n$;*

(c) *the discriminant of $K/\mathbb{Q}$ is divisible by n^n;*

(d) $|\mu_K| = 2^a$ *for a certain $a \in \mathbb{Z}$, $a > 0$; the number 2^{a-1} divides n, but is not equal to it;*

(e) $(\mathscr{O}, \mathrm{tr}_{K,\theta}/m)$ *is isometric to a root lattice $\mathbb{Z}^{2^{a-1}} \oplus L$, where L is a nonzero even root lattice whose rank is divisible by 2^{a-1}, and $\mu_K = \{x \in \mathscr{O} \mid (\mathrm{tr}_{K,\theta}/m)(x, x) = 1\}$;*

Quadratic fields K with lattice $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ similar to a root lattice of mixed type

Theorem

If K is a quadratic (i.e., $n = 2$) field, then the following two properties of a pair K, θ are equivalent:

(a) $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ *is similar to a root lattice of mixed type;*

(b) *either K is isomorphic to $\mathbb{Q}(\sqrt{2})$ and $\theta = \mathrm{id}$, or K is isomorphic to $\mathbb{Q}(\sqrt{-2})$ and θ is the complex conjugation.*

If (a), (b) *hold, then $(\mathscr{O}, \mathrm{tr}_{K,\theta}/2)$ is isometric to the lattice $\mathbb{Z}^1 \oplus \mathbb{A}_1$.*

Cubic fields K with lattice $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ similar to a root lattice of mixed type

Theorem

If K is a cubic (i.e., $n = 3$) totally real field and $\theta = \mathrm{id}$, then the following two properties of a pair K, θ are equivalent:

(a) $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ *is similar to a root lattice of mixed type;*

(b) K *is the maximal totally real subfield of a 9th cyclotomic field.*

If (a), (b) *hold, then* $(\mathscr{O}, \mathrm{tr}_{K,\theta}/3)$ *is isometric to the root lattice* $\mathbb{Z}^1 \oplus \mathbb{A}_2$.

Lattices $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ similar to root lattices of mixed type: Examples

The following group of examples gives an infinite series of pairs K, θ, for which $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ is similar to a root lattice of mixed type.

The construction uses cyclotomic fields $\mathbb{Q}(\zeta_d)$, where ζ_d is a primitive root of 1 of degree d, and their maximal totally real subfields

$$\mathbb{Q}(\zeta_d)^+ := \mathbb{Q}(\zeta_d + \zeta_d^{-1}).$$

Lattices $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ similar to root lattices of mixed type: Examples

Example (A. A. Andrade and J. C. Interlando)

Let $a \in \mathbb{Z}$, $a > 2$ and let

$$K = \mathbb{Q}(\zeta_{2^a})^+, \quad \theta = \mathrm{id}.$$

Then

$$m = n = 2^{a-2} \quad \text{and}$$

$(\mathscr{O}, \mathrm{tr}_{K,\theta}/m)$ is <u>isometric to the root lattice</u> $\mathbb{Z}^1 \oplus \mathbb{A}_1^{2^{a-2}-1}$.

Lattices $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ similar to root lattices of mixed type: Examples

Example (E. Bayer-Fluckiger)

Let $b \in \mathbb{Z}$, $b > 1$ and let

$$K = \mathbb{Q}(\zeta_{3^b})^+, \quad \theta = \mathrm{id}.$$

Then

$$m = n = 3^{b-1} \quad \text{and}$$

$$(\mathscr{O}, \mathrm{tr}_{K,\theta}/m) \ \text{ is isometric to the root lattice } \mathbb{Z}^1 \oplus \mathbb{A}_2^{(3^{b-1}-1)/2}.$$

Lattices $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ similar to root lattices of mixed type: Examples

Example (E. Bayer-Fluckiger and P. Maciak)

Let $a \in \mathbb{Z}$, $a > 2$ and let

$$K = \mathbb{Q}(\zeta_{2^a} - \zeta_{2^a}^{-1}) \subset \mathbb{Q}(\zeta_{2^a}),$$
$$\theta \text{ is the complex conjugation}$$

(this field K is a purely imaginary quadratic extension of the totally real field $\mathbb{Q}(\zeta_{2^{a-1}})^+$). Then

$$m = n = 2^{a-2} \quad \text{and}$$

$(\mathscr{O}, \mathrm{tr}_{K,\theta}/m)$ is <u>isometric to the root lattice</u> $\mathbb{Z}^1 \oplus \mathbb{A}_1^{2^{a-2}-1}$.

Lattices $(\mathscr{O}, \mathrm{tr}_{K,\theta})$ similar to root lattices of mixed type: Examples

Example

Let $a, b \in \mathbb{Z}$, $a > 1$, $b > 1$ and let

$$K = \mathbb{Q}(\zeta_{2^a}) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{3^b})^+ = \mathbb{Q}(\zeta_{2^a}(\zeta_{3^b} + \zeta_{3^b}^{-1})) \subset \mathbb{Q}(\zeta_{2^a 3^b}),$$

θ is the complex conjugation.

Then

$$m = n = 2^{a-1}3^{b-1} \quad \text{and}$$

$(\mathscr{O}, \mathrm{tr}_{K,\theta}/m)$ is <u>isometric to the root lattice</u> $\mathbb{Z}^{2^{a-1}} \oplus \mathbb{A}_2^{2^{a-2}(3^{b-1}-1)}$.